NORMAL CHARACTERIZATIONS
VIA THE SQUARES OF RANDOM VARIABLES

by

Seymour Geisser

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University of Minnesota
Minneapolis, Minnesota

1. Introduction.

In a recent paper by Roberts [1], in his section entitled "Characterization Results," he conjectured as a generalization of a previous result of Roberts and Geisser [2]: "If X and Y are independent and identically distributed then X^2, Y^2 and $W^2 = (aX + bY)^2/(a^2+ b^2)$ are all $\chi^2(1)$ for some $a, b \neq 0$ if and only if X and Y are standard normal."

After making this conjecture he said "but this does not seem to be a correct statement, although no counterexample is presented." We shall show here that the conjecture is actually correct so that no counterexample is possible. In doing so we shall also strengthen some of the results in both of the aforementioned papers.

2. Results.

We start from the following which is given as Corollary 1 of [1] and is an immediate consequence of Corollary 1 of [2].

Corollary 1.

The random variable W^2 is $\chi^2(1)$ if, and only if, the characteristic function of W satisfies

$$\varphi_W(t) + \varphi_W(-t) = 2 \exp(-t^2/2).$$

We now prove a stronger version of Corollary 2 of [2].

Corollary 2.

Let X be $N(0, 1)$ and independent of Y. Then $W^2 = (aX + bY)^2/(a^2+ b^2)$ for some $a, b \neq 0$ is $\chi^2(1)$ if, and only if, Y^2 is $\chi^2(1)$.

Proof:

When W^2 is $\chi^2(1)$ this implies by Corollary 1 that

$$\varphi_W(t) + \varphi_W(-t) = 2e^{-t^2/2}.$$

Further

(1) $$\varphi_W(t) = \exp[-a^2t^2/2(a^2+b^2)]\varphi_Y(bt/\sqrt{a^2+b^2}).$$

Hence

$$2\exp(-\frac{t^2}{2}) = [\varphi_Y(bt/\sqrt{a^2+b^2}) + \varphi_Y(-bt/\sqrt{a^2+b^2})]\exp[-a^2t^2/2(a^2+b^2)].$$

Let $u = bt/\sqrt{a^2+b^2}$ and solving the above we obtain

$$2e^{-u^2/2} = \varphi_Y(u) + \varphi_Y(-u).$$

Thus Y^2 is $\chi^2(1)$ by Corollary 1. Conversely let Y^2 be $\chi^2(1)$. Then the characteristic function of Y satisfies

$$\varphi_Y(u) + \varphi_Y(-u) = 2\exp[-u^2/2].$$

Then let $u = bt/\sqrt{a^2+b^2}$ in (1) above. Hence we obtain easily

$$\varphi_W(t) + \varphi_W(-t) = 2e^{-t^2/2}$$

thus completing the proof. We now prove the following:

<u>Corollary 3</u>.

Let X and Y be independent with X^2 and Y^2 each distributed as $\chi^2(1)$. Then $W^2 = (aX + bY)^2/(a^2+b^2)$ is $\chi^2(1)$ for some $a, b \neq 0$ if, and only if, at least one of the pair X, Y is $N(0, 1)$.

<u>Proof</u>:

First we let X be $N(0, 1)$ and Y^2 be $\chi^2(1)$. Then, by Corollary 2, W^2 is $\chi^2(1)$. Conversely, if W^2 is $\chi^2(1)$, then

(2) $$\varphi_W(u) + \varphi_W(-u) = \varphi_X(u) + \varphi_X(-u) = \varphi_Y(u) + \varphi_Y(-u) = 2\exp[-u^2/2].$$

Further

(3) $$\varphi_W(t) = \varphi_X(at/\sqrt{a^2+b^2})\varphi_Y(bt/\sqrt{a^2+b^2}).$$

Hence from (2) and (3) we obtain

(4) $$2\exp[-t^2/2] = \varphi_X(at/\sqrt{a^2+b^2})\varphi_Y(bt/\sqrt{a^2+b^2})+\varphi_X(-at/\sqrt{a^2+b^2})\varphi_Y(-bt/\sqrt{a^2+b^2}).$$

Further, from (2) and (4) we obtain

$$(5)\qquad 4\exp[-t^2/2] = [\varphi_X(at/\sqrt{a^2+b^2})+\varphi_X(-at/\sqrt{a^2+b^2})][\varphi_Y(bt/\sqrt{a^2+b^2})+\varphi_Y(-bt/\sqrt{a^2+b^2})].$$

Multiplying out the r.h.s. of (5) and applying (4) we obtain

$$(6)\qquad 2\exp[-t^2/2] = \varphi_X(-at/\sqrt{a^2+b^2})\varphi_Y(bt/\sqrt{a^2+b^2}) + \varphi_Y(-bt/\sqrt{a^2+b^2})\varphi_X(at/\sqrt{a^2+b^2}).$$

Subtraction of (6) from (4) yields

$$(7)\qquad [\varphi_Y(bt/\sqrt{a^2+b^2}) - \varphi_Y(-bt/\sqrt{a^2+b^2})][\varphi_X(at/\sqrt{a^2+b^2}) - \varphi_X(-at/\sqrt{a^2+b^2})] = 0.$$

Hence at least one of the factors on the l.h.s. of (7) must be zero. Therefore say

$$\varphi_X(at/\sqrt{a^2+b^2}) = \varphi_X(-at/\sqrt{a^2+b^2}),$$

and from (2) we see that

$$2\varphi_X(at/\sqrt{a^2+b^2}) = 2\exp[-a^2t^2/2(a^2+b^2)]$$

and X is then $N(0, 1)$. Reversing the roles of X and Y obviously leads to the same conclusion.

Now we are in a position to prove the Roberts conjecture by noting first that if X and Y are $N(0, 1)$, then clearly X^2, Y^2 and W^2 are $\chi^2(1)$. Conversely if X and Y are identically distributed such that X^2, Y^2 and W^2 are all $\chi^2(1)$, then by Corollary 3 at least one of the pair X, Y is $N(0, 1)$. Hence, both are since they are assumed to be identically distributed.

Further, we now are also in a position to extend Theorem 5 of Roberts to the following:

<u>Theorem.</u>

Let X and Y be independent and identically distributed. Then X and Y are $N(0, 1)$ if, and only if $W_1^2 = (aX + bY)^2/(a^2+ b^2)$ and

$W_2^2 = (aX - bY)^2/(a^2 + b^2)$ for some $a, b \neq 0$ are $\chi^2(1)$. (For Roberts' Theorem 5, set $a = b = 1$).

Proof:

We note that the "only if" part is trivial. Further from the fact that W_1^2 and W_2^2 are both $\chi^2(1)$ we obtain

$$4 \exp[-t^2/2] = [\varphi_X(at/\sqrt{a^2+b^2}) + \varphi(-at/\sqrt{a^2+b^2})][\varphi_Y(bt/\sqrt{a^2+b^2}) + \varphi_Y(-bt/\sqrt{a^2+b^2})]$$

Let $u = at/\sqrt{a^2+b^2}$ so that

$$4 \exp[-u^2/2]\exp[-u^2b^2/2a^2] = [\varphi_X(u) + \varphi_X(u)][\varphi_Y(bu/a) + \varphi_Y(-bu/a)].$$

Let $2\tau(u) = \varphi_X(u) + \varphi_X(-u)$. Since φ_X and φ_Y are the same (X, Y being identically distributed), we have then by equating factors that X^2 and Y^2 are both $\chi^2(1)$. Now applying the Roberts conjecture previously proven the theorem is obtained.

3. A New Conjecture.

At this point a further conjecture is in order. We now conjecture the following: If X and Y are i.i.d., then $W^2 = \frac{1}{2}(X + Y)^2$ is $\chi^2(1)$, if, and only if, X and Y are standard normal.

Of course the sufficiency part is trivial. To prove necessity essentially requires that there exist only one characteristic function $\varphi(t)$ which is a solution to the equation

$$\varphi^2(t) + \varphi^2(-t) = 2e^{-t^2}$$

and that of course be $\varphi(t) = e^{-t^2/2}$.

To establish the truth or falsity of this appears to be difficult. If this conjecture is true the slight complication of extending it for some $a, b \neq 0$ does not appear to be consequential.

References

[1] Roberts, Charles (1971). On the distribution of random variables whose m-th absolute power is gamma. Sankhyā 33 (2) 229-232.

[2] Roberts, Charles and Geisser, Seymour (1966). A necessary and sufficient condition for the square of a random variable to be gamma. Biometrika 53 275-277.